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A Learning Support System for Mathematics with Visualization of Errors in Symbolic Expression by mapping to Graphical Expression

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Abstract: It is difficult for mathematics learners to solve problems that require describing solution procedures. This issue was revealed in the PISA and TIMSS tests. This issue is possibly caused by a lack of understanding of the relation between symbolic and graphical representations in mathematics. This work proposes a learning support system that realizes understanding of the relation between both types of expression. This paper describes the development of our system and reports on its effectiveness and considerations. We realize a function that converts symbolic expressions entered by learners to graphical representations, and a function for manipulating the converted graphic. The conversion function visualizes an input symbolic expression as a graphic. If the symbolic expression contains an error, the function visualizes the error so that learners can become aware of it. This is “learning from errors” by error visualization. The function also realizes learner operations related to the visualized graphic. The operation range is limited by constraints in the symbolic sentence input by the learner. Through their operations, learners deepen their understanding of how the symbolic sentence influences the graphic and clarify their understanding of the relation between the symbolic expression and its graphical representation. An experimental test verified that the proposed system using this method is effective in mathematical learning and facilitates learner understanding of the contents of symbolic sentences.

Keywords: Mathematics education, error visualization, learning by error, learning support system

1. Introduction

This paper describes development of a learning support system for understanding mathematics through expression transformation and active operations.

One problem in mathematics, revealed in the PISA and TIMSS tests, is that learners cannot describe their own ideas. One reason for this is learners trying to improve the efficiency of problem solving by simply memorizing mathematical formulas, which they apply in tests and tasks. Doing so has only temporary value and does not address the root of the problem. This is not “learning,” because there is a high possibility that the learner will retain an immature approach when thinking about other problems. In mathematics, learners must understand symbolic expressions as both a mathematical statement and a graphic expression of what it means. Many learners do not understand the meaning of their answers, which are presented as procedural flows. The ability to solve problems is necessary, but more important is knowing the meaning of the procedure applied. This requires understanding the relation between symbolic expressions and graphical representations. It is important to understand the graphical meaning of problem sentences and learner solutions. If learners provide incorrect answers, they should be able to see their error immediately. One method of doing so is converting a mathematical statement (hereinafter, a "sentence") into a graphic, through which learners can discover errors in their answers and see why they are wrong.

Many works have suggested that this "learning from errors" approach plays an important role in knowledge correction and understanding, particularly when learners notice their errors themselves (Perkinson, 2000; Hirashima, 2004; Hirashima & Horiguchi, 2004; Tomoto, Imai, Horiguchi & Hirashima, 2013). Recent works have suggested error visualization as a way of grasping action and reaction dynamics (Horiguchi & Hirashima, 2001; Horiguchi & Hirashima, 2002; Imai, Tomoto, Horiguchi & Hirashima, 2008), inference error visualization in geometric proofs (Funaoi, Kameda, & Hirashima, 2009), visualizing errors as 3D models (Matsuda et al., 2008), and visualization of errors in English composition using animation (Kunichika et al., 2008). To produce "intrinsic awareness" of errors, it is effective to indicate what kind of conclusion results from learner answers and to show the learner contradictions that arise.

There are few support systems that can handle geometry problems in high-school mathematics; such support systems are generally targeted at geometric proofs in junior high-school mathematics. In geometric proofs studied in high school, it is more difficult to understand relations between symbols and graphs. Such systems are thus likely necessary to support mathematical learning in both primary education and higher education.

Our work aims at developing learning support systems for learners who struggle when writing answers and those who cannot understand presented solutions. We aim to help learners understand the relation between symbolic mathematical expressions and graphical representations, and aim for improvement of academic mathematical ability.

The system converts symbolic expressions into graphical representations. The system prepares sentences corresponding to the problem, learners select the sentences closest to their answers, and the system graphically shows how the selected sentence changes the system. We expect that learners will perceive their errors through the conversion of their symbolic expressions into graphical representations and by manipulating the diagram. Furthermore, sentences corresponding to each line in the solution allow grasping individual constraints in the figure. Through active manipulations, learners can learn whether operations produce a figure like that imagined as the solution. We can visualize symbolic sentences selected by the learner as figures and by adding operations to the figure we can support "intrinsic awareness" of what was imagined or the unintended results of the actions. We also had a goal of scoring overall answers in a way that shows what the graphical representation of each line in the answer sentence means. In the proposed method, we verify learning effectiveness through the developed system and paper tests.

Test scores from before and after using the system for learning showed that there was a statistically significant improvement from use of the system. The results indicate that the system is effective for mathematical learning, and that the transformation of expressions and the manipulation of figures are effective for building mathematical understanding.

The remainder of this paper is organized as follows. Section 2 describes the effects of mathematical expressions, and section 3 describes the conversion of expressions. Section 4 describes the proposed system, section 5 provides an evaluation of the experiment, and section 6 comments on its results. We close in section 7.

2. On the effect of mathematical expressions

Nakahara (1995) proposes five categories of mathematical expressions: realistic, operational, graphical, linguistic, and symbolic. These expressions are implemented in mathematics lessons as expressions, figures, tables, and graphs, and can deepen learners' understanding of mathematical concepts. Furthermore, because expressions are learning goals in the system, it is necessary to organize and use representation methods. In this paper, we focus on symbolic and graphic expressions in mathematics because they are used in most mathematical learning and are thus presumably the most important for understanding mathematics. Certainly, learners must be able to use both. We propose a method for visualizing the mathematical situation and promoting learner understanding. Specifically, the proposed method improves learners' ability to construct solutions to mathematical problems by producing graphical figures corresponding to their mathematical statements.

Recent studies on graphical representations have indicated that they have multiple roles and effects. For example, they can lessen the role of working memory in children learning mathematics, produce concrete models, make it easier to find related information, and make features of the problem clearer (Van Essen & Hamaker, 1990). Furthermore, graphical representations more clearly express problem structures, provide a basis for correctly solving problems, allow tracing learners' information knowledge, and clearly show implicit information (Diezmann & English, 2001). However, despite research themes aimed at promoting the understanding of symbolic expressions by exploiting graphical representation, there has been no change in the present situation, in which graphic expressions are not used well.

There have been many studies focusing on graphical representations in mathematics education. Thereby, effects by graphical expression have also been clarified (Hiroi, 2003; Nunokawa, 1993; Doishita, 1986). We examined whether graphical representations can promote understanding of mathematics based on symbolic expressions. Materials used by learners mainly pose problems as statements in the form of symbolic expressions. For solving problems, we believe it is important that learners consider what figures can be made from symbolic expressions, rather than starting with a symbolic expression after seeing the nature of the figure. This paper thus focuses on symbolic expressions to approach the above problem, observing how symbolic sentences written by the learner affect the corresponding graphic. In doing so, we aim to understand the relation between sentences and graphics.

3. Conversion of expressions

One factor causing learners to struggle with problems is that they cannot grasp quantitative relations in sentences. To address this problem, it is important to use various modes of expression. One factor causing learners to struggle with problems is that they cannot grasp quantitative relations in sentences. To address this problem, it is important to use various modes of expression. Especially, symbolic and graphical representation is heavily used, so they are important representations.

Therefore, we propose a learning method for understanding relations between symbolic and graphical representations. As a solution, we use "expression conversions." Using this transformation of expressions and "graphical operations," described below, we support learners' endogenous awareness and improve mathematical understanding. Figure 1 shows an overview of the method.

To clarify the relation between symbolic and graphical expressions, we explain the conversion from symbolic representations to graphical representations and the visualization of errors.

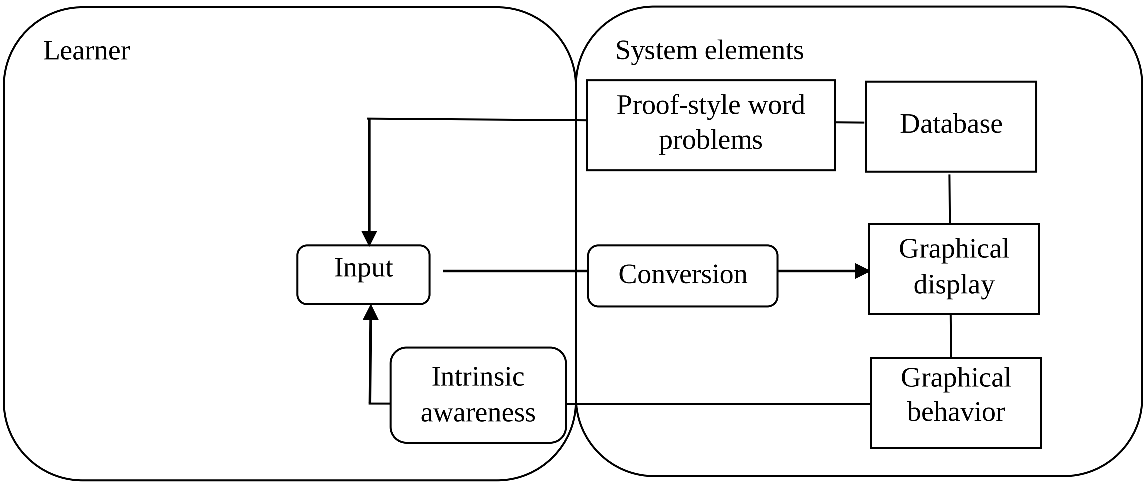

Figure 1. Transformation model

3.1. Conversion from symbolic to graphical representation

When converting from symbolic to graphical representations, it is difficult for the system to decode content entered as natural language and convert it to a graphical representation. The proposed system therefore adopts an answer format that uses solution templates with short sentence prepared from symbolic expressions. Here, we define “short sentences” as those that can be drawn as a graphic from one element in the sentence. An example would be a sentence containing “a statement defining a point” and “a relation of that point.” In this case, the point definition becomes one element and is made visible in the graphic. Here, the point definition is given as point coordinates (x, y), and the point relationship is a sentence such as “AP = BP,” which stipulates the line segments AP and BP are of equal length. This paper calls this type of statement a short sentence. The above constructs are used in multiple answer selections for each problem, as shown in Fig. 2. At this time, answers are input in advance in a database and recalled as necessary to draw the graphic. The graphic can be manipulated within a range of constraints specified from the sentences. By manipulating graphics created by the learner, the system supports endogenous awareness. By manipulating the graphic, learners can experience and check constraints themselves while viewing differences between the resulting graphic and the actual answer, thereby correcting learner errors.

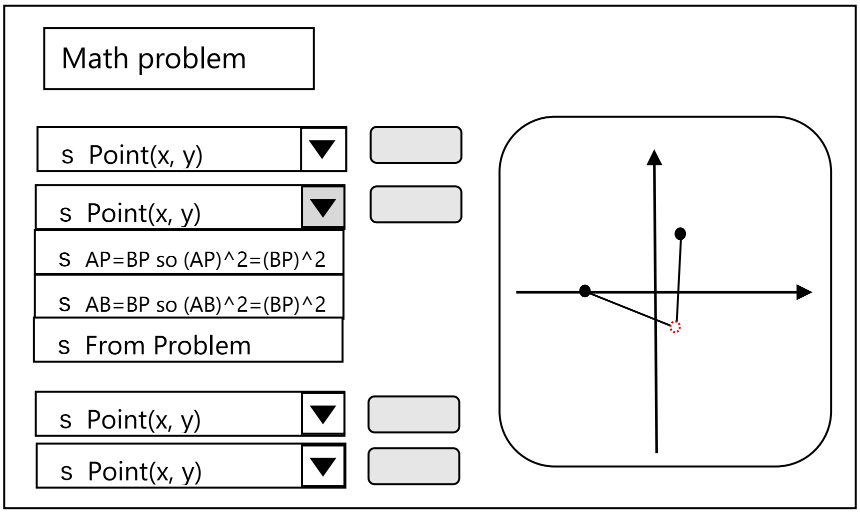

Figure 2. Answer template usage example

3.2. Conversion from graphical to symbolic representations

Many learners struggling with mathematics, as well as those with some knowledge, cannot well explain the characteristics of mathematical figures. As mentioned in section 1, few students, even highly scoring ones, can describe their thoughts and sentences. One reason for this is that learners understand neither the construction of the graphic nor the meaning of the problem. In other words, these learners do not clearly grasp the correspondence between symbolic and graphical representations. Conversion from figures to symbols is also important to solving this problem.

4. The proposed system

Using the method described in section 3, we designed and developed a system aimed at improving understanding of mathematics. The system converts symbolic expressions selected by the learner into a graphical representation and makes the learner observe and manipulate correct and erroneous graphs. The problem range becomes, for example, the “trajectory” of the point given certain constraints. In section 6, we will examine the effect of this method on learning in the range of the trajectory for that type of problem. When constructing an answer, the system provides learners with short sentences, namely those composed of few elements. Learners then use the elements of these short sentences as answer templates used to solve the problem. Developed functions and system operations are limited to graphical functions using short sentence symbols, functions for manipulating the figure, and a true/false judgment function. Functions are flexibly applied to all problems in the system. Especially since questions in this system start from assumptions of the positions of points, functioning is highly versatile. For example, when a learner uses a template to generate points while solving a problem, the points can be moved (manipulated) over the entire plane if their location is determined by an unknown quantity. If one axis is determined by a constant, the point can be moved along that axis. Questions with this kind of function have been created in the system. As described above, the system can produce graphics matching the problem contents, based on answer templates selected by the learner. Furthermore, learners can manipulate the generated points and lines. However, if there are drawing conditions (such as placing a point on the x-axis) specific in the selected answer template, the drawing can be manipulated only in ways that continue to satisfy that constraint.

In mathematics solutions, we use point (x, y), (x, 0) answer templates for manipulating points. The system allows confirming what kind of expression and movement learner-provided sentences will produce. As the answer construction progresses, the drawing area is gradually limited. For example, after setting a point P to (x, y) and adding the condition AP = BP, the point is constrained by that equation. In another pattern, setting point P to (x, 0) fixes the point, so it cannot be moved at all. Learners can observe and experience the situation when partial answers are mistaken. They can also experience such things through the operation of points, and can approach the relation between symbolic expressions and their graphical representation.

The goal of this system is to understand the contents of the diagram as expressed by the meaning and restrictions of learner-provided sentences.

The system was developed using the Visual Basic 2010 programming language.

4.1. Activities in the system

The operational flow of the system is as follows.

The system presents three problems, and learners select the problem they wish to solve. They are then taken to a “Practice” screen that presents a tutorial for system operation. Other problems are for calculating coordinates of a fixed point and a point P satisfying a certain condition, and a “trajectory” problem for calculating the point on which a straight line exists, where the point is not fixed. Our goal is for the learner to be able to solve the “trajectory” problem after using the system.

Learner activities within the system are as follows:

1. Check the mathematical problem provided by the system and select the answer that the learner intends as optimal from the given multiple symbolic mathematical expressions.
2. The system visualizes the selected sentence. The learner observes the graphic, understands the numerical content of the symbolic sentence and the relation with the sentences before and after it, clearly showing the relation between the mathematical sentence and the graphic.
3. In the visualized figure, it is possible for the subject to manipulate the graphic (move the generated point with the mouse). Users can thus experience constraints included in sentences, based on visualization from manipulation of the resulting graphic or the sentence while viewing how the completed answer sentence influences the graphic.

4.2. Interface

In the mathematical solution method, learners need to construct sentences according to the problem. An inherent problem is that even if the student represents the contents of the sentence as a diagram and attempts to understand it, the amount of work increases. As a result, the meaning of the sentence cannot be grasped, and answers may be erroneous. There may also be learners who do not understand which mathematical symbol sentences should be used to find the solution. As a support system, we provide some answer sentences, and the learner selects an answer from among them.

As described above, even for learners who cannot provide an answer, the system prepares answer templates (typical sentences in which sentences are built using mathematical expressions) and selects a symbolic expression from among those answers.

Within the system, it is possible to select an answer via a pulldown menu. By clicking a button corresponding to a line, a figure reflecting that sentence is generated in a frame on the right side of the screen. Based on the quantitative constraints contained in the selected sentence, the generated figure can be manipulated within the scope of the constraint. A learner who gives up without being able to provide a solution is provided with intrinsic awareness that “using this sentence for this problem will produce a figure like that in the system.” Learners can expect effects that will allow them to reflect their way of thinking about problems that have not been addressed so far.

Unless the solution is logically wrong, errors can be positively visualized to allow understanding why it is an error. The figure shows an operation example when the point is set as (x, 0). As an example in which the system cannot visualize the learner’s symbolic text as a diagram, consider a case where a point P is defined on the first line of the solution and a point B on a second line, yet it is declared that AP = BP on a third line. In this case, since the number of points does not match and point A is not defined, the image cannot be drawn. The system shows the user the answer corresponding to their input. By manipulating the drawing, learners can confirm differences between solutions they imagined, and if the answer was wrong they can reconstruct the sentence and confirm. Further, when providing a solution, they can explain their answers in the lower left of the system screen.

4.3. Answer diagnosis method

The list of symbolic expression answers is obtained by previously calling up values stored in a database. For each table in the database, symbolic expressions are divided into points, lines, expressions, and so on. Within that table we further store problems and the attributes of each sentence. We perform diagnoses based on information in this database. Assume that the correct answers for the first through third lines are stored in the temporary variables a1, b1, and c1, respectively, so correct answers are provided when the learner clicks the answer buttons for a1, b1, c1. Even if incorrect answers are provided, the system analyzes the incorrect answer pattern and generates a graphic corresponding to the process described by the provided sentences. In the diagnostic result, the system converts the sentence into a graphic if a logically correct symbolic structure is formed. If the meaning is illogical, drawing is not performed and learners are provided with an explanation to that effect in the message and status pane.

5. Evaluation Experiment

5.1. Purpose

To evaluate the extent to which the proposed system can contribute to learners’ mathematical understanding, we conducted experiments as described below to evaluate the expression conversion function and graphic manipulation functions. The results and consideration are described later.

5.2. Method

To confirm the effectiveness of system functions, we also used a system without an expression conversion function. Subjects were 18 university students who had taken at least one mathematics course intended for those in math-heavy fields of study. Although this work covers the scope of high school mathematics, college students do not fully understand the relationship between symbols and graphics even in mathematics in the high school range. Before using the system, the experimental procedure was explained to the subjects. The system was operated by an author of the study, and the method and operation of the system was described, including a tutorial on how problems are handled in the case of an incorrect answer.

The experiment procedure was preliminary system testing before learning (10 minutes) and the first half of system learning (30 minutes), followed by post-test 1 (10 minutes). Subsequently, the latter half of system learning (30 minutes) was followed by post-test 2 (10 minutes). Subjects answered a questionnaire after completing the experiment. The 18 subjects were divided into 2 groups, group A (learning in the first half: proposed system; learning in the second half: no conversion function) and group B (learning in reverse order to group A). The pre-test, post-test 1, and post-test 2 were all the same problem. Question contents were taken from an 11th-grade mathematics textbook, and involved topics such as distance and trajectory between two points. In big question 1, subjects read a short sentence for each small question and drew a diagram. In big question 2, subjects determined the presence or absence of inconsistencies between sentences and diagrams. In big question 3, subjects calculated point coordinates and found trajectories. Only big question 3 presented a problem in the format presented in the mathematical textbook. The subjects answered 11 questions covering the above content on paper. However, for big question 3, we established different evaluation criteria: the correctness of each line of the answer sentences produced by subjects was evaluated.

5.3. Provisional

The expected effects of this experiment on learners is as follows:

- Expression conversion from symbols to figures is effective for mathematical understanding.
- Manipulating graphics leads to understanding of the source symbolic sentence.

5.4. Results

Tables 1–6 show the scores and test results for pre- and post-test 1 and post-test 2 for groups A and B obtained by the experiment described in section 5. All tests were evaluated with a significance level of $p<.01$.

Question 3, which did not improve in this test, checked the extent to which answers could be described before and after system learning. We confirmed correct answers for each sentence by the reevaluation method. Tables 5–8 show scores for the small questions and the results of analysis of variance. Table 1 shows the average number of correct test answers for each group. Table 2 shows the results of analysis of variance on the number of correct answers from each of the three test results in the three-step test. Table 4 shows the results of performing a subordinate test (multiple comparison within individuals) to see in detail at which timing significant differences appeared. Tables 5–8 show the analysis results for large question 3.

From Table 1, the average number of correct answers immediately after system learning in group A (from the first to second post-testing) improved by 1.9. In Group B, the average number of correct answers immediately after system learning (from first to second post-testing) improved by 1.8. This shows that system learning outperforms the average number of correct answers as compared to using the system without the function for converting symbolic representations to graphic representations. Furthermore, the results of variance analysis in Table 2 confirm significant differences in individuals in both groups. Table 3 shows that the tests in which significant differences were observed in group A are the results of the pre-test and post-test 2, and the results of post-test 1 and post-test 2. The tests with significant differences in group B are found to be the results of the pre-test and post-test 2, and of post-test 1 and post-test 2.

From the above, the system had a learning effect in both groups, while there was no significant difference when using the system without graphic conversion of expressions.

From the results of the questionnaire, high evaluations were gained on the importance of understanding the relation between sentences and figures and the expression transformation and graphic manipulation functions.

This suggests that learning using this system (with the expression transformation function) is effective in mathematical learning.

Table 1. Average number of correct answers for group A and group B

A	Pre-test	Post-test 1	Pro-test 2	B	Pre-test	Post-test 1	Pro-test 2
Big question 1	2.0	2.4	2.9	Big question 1	2.1	3.4	3.6
Big question 2	2.6	2.8	3.7	Big question 2	3.0	3.3	3.7
Big question 3	0	0	0.6	Big question 3	0	0.1	0.3
Total	4.6	5.2	7.1	Total	5.1	6.9	7.6

Table 2. Analysis of variance of the 3-step test

A	SS	df	MS	F	p	B	SS	df	MS	F	p
Test	31.63	2	13.62	18.47	p<.01	test	28.74	2	14.37	11.17	p<.01

Table 3. Ryan's method (left: group A; right: group B)

Pair	R	level	t	p	Pair	r	level	t	p
1-3	3	0.03	5.86	p<.01	1-3	3	0.03	4.57	p<.01
2-3	2	0.07	4.33	p<.01	2-3	2	0.07	1.24	n.s.
1-2	2	0.07	1.53	p<.01	1-2	2	0.07	3.32	p<.01

Table 4. Average number of correct answers for big problem 3

A	Pre-test	Post-test 1	Pro-test 2	B	Pre-test	Post-test 1	Pro-test 2
Small question 1	0.7	2.4	4.3	Small question 1	0.6	3.0	4.2
Small question 2	0.4	0.8	1.8	Small question 2	0.3	0.7	1.9
Total	1.1	3.2	6.1	Total	0.9	3.7	6.1

Table 5. Analysis of variance for big question 3 (1)

A	SS	df	MS	F	p	B	SS	df	MS	F	p
Test	60.52	2	30.26	18.86	p<.01	test	62.74	2	31.37	20.98	p<.01

Table 6. Ryan's method for big question 3 (1) (left: group A; right: group B)

Pair	R	level	t	p	Pair	r	level	t	p
1-3	3	0.03	4.87	p<.01	1-3	3	0.03	6.36	p<.01
2-3	2	0.07	2.50	p<.01	2-3	2	0.07	2.12	p<.01
1-2	2	0.07	2.36	p<.01	1-2	2	0.07	4.24	p<.01

Table 7. Analysis of variance for big question 3 (2)

A	SS	df	MS	F	p	B	SS	df	MS	F	p
test	8.67	2	4.33	8.00	p<.01	test	12.07	2	6.03	4.40	p<.01

Table 8. Ryan’s method for big question 3 (2) (left: group A; right: group B)

Pair	R	level	t	p	Pair	r	level	t	p
1-3	3	0.03	3.84	p<.01	1-3	3	0.03	2.81	p<.01
2-3	2	0.07	2.89	p<.01	2-3	2	0.07	2.22	p<.01
1-2	2	0.07	0.96	n.s.	1-2	2	0.07	0.60	n.s.

6. Discussion

This work proposed development of a learning support system for learners who are not good at constructing answers and do not understand solutions based on elucidation from symbolic expressions and graphical representations. We demonstrated that the proposed system is suitable for mathematical learning. Pre- and post-testing showed that subjects could produce answers from problem solving, along with graphic answers. We also confirmed that points of confusion at the time of testing were corrected after learning in the system. The above suggests that learners themselves experienced expression transformation to diagrams and manipulation of figures appeared as an effect.

7. Summary

We proposed transformation of mathematical expressions and graphic manipulations as a method for improving learner understanding of mathematics. The selected target expressions were symbolic and graphic, two forms of expression that are important for understanding mathematics. Using the proposed system, we conducted experiments to verify its effectiveness at improving mathematic understanding. The proposed method of converting mathematical expressions promoted learner understanding, as evidenced from experimental results. Furthermore, by manipulating graphics, we were able to support understanding of motion constraints and quantity relations in graphics included along with symbolic sentences.

Our findings are summarized as follows:

1. At the time of learning, there were significant differences in test results when the system had functions for converting sentences into graphics. This demonstrates that this method is suitable for mathematical learning.
2. Learner efforts to correct errors were indicated by diagram manipulations. In immediate post-tests, learners presented their own ideas by drawing graphics.
3. From the results of a questionnaire, students were able to test their answers by converting symbols to graphics in the system, allowing awareness of the importance of converting mathematical expressions. Also, we found that manipulation of figures leads to discovery of errors.

In future work, we will improve the system based on the result of this time.

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References

Carmel M.Diezmann & Lyn D.English. (2001). The Roles of Representation in School Mathematics, (NCTM2001YEARBOOK), NCTM, *Promoting the Use of Diagrams as Tools for Thinking, 77-89.*

Doishita, A., Shimizu, H., Ueoka, T. & Ichisaki, M. Teaching Strategy in the Problem Solving : -Conducting a Survey of the Pupils on Pictures and Figures and Putting it into Practice-. *Japan Society of Mathematical Education-Math education-*, 68(4), 18-22. (in Japanense)

Funaoi, H., Kameda, T., & Hirashima, T.(2009). Visualization of an Error in Solution of Geometry Proof Problems. *Japan Journal of Education Technologys*,32(4), 425-433. (in Japanese)

Hirashima, T. (2004). Aiming at Interaction Giving Awareness of error. *Human Interface: The Transaction of Human Interface Society*, 6(2), 99-102. (in Japanese)

Hirashima, T., & Horiguchi, T. (2004). Attempt to Visualize Errors from Orienting Learning from Error. *Transactions of Japanese Society for Information and Systems in Education*, 21(3), 178-186. (in Japanese)

Hiroi, H.(2003). Understanding problems by graphic in elementary school fifth grade. *Japan Society of Mathematical Education-Arithmetic/ Mathematics-*, 85(6), 10-19. (in Japanese)

Horiguchi, T., & Hirashima, T. (2001). Simulation-Based Learning Environment for Assisting Error-Awareness - Management of Error-Based Simulation Considering the Expressiveness and Effectiveness-. *Transactions of Japanese Society for Information and Systems in Education*,18(3),364-376. (in Japanese)

Horiguchi, T., & Hirashima, T.(2002). Simulation-Based Learning Enviroment for Assisting Error-Correction. *The Japanese Society for Artificial Intelligence*, 17(4), 462-472. (in Japanese)

Imai, I., Tomoto. T, Horiguchi, T., & Hirashima, T.(2008). A classroom practice of error-based simulation to improve pulils' understanding of mechanics: the “challenge to Newton!” project. *Transactions of Japanese Society for Information and Systems in Education*, 25(2),194-203. (in Japanese)

Kunichika, H., Koga, T., Deyama, T., Murakami, T., Hirashima, T., & Takeuchi, A.(2008). Learning Support for English Composition with Error Visualization. *The IEICE transactions on information and systems*, 91(2), 210-219. (in Japanese)

Matsuda, N., Takagi, S., Soga, M., Horiguchi, T., Hirashima, T., Taki, H., & Yoshimoto, H.(2008). Error Visualization for Pencil Drawing with Three-Dimensional Model. *The IEICE Transactions on Information and Systems*, 91(2), 324-332. (in Japanese)

Nakahara, T. (1995). *Study of constitutive approach in mathematics and mathematics*. seibunsha, 389.

Nunokawa, K.(1993). Role of graphic in mathematical problem solution and meaning by solver. Tatsuro Miwa's Retirement Commemorative Papers Collection Committee-Advancement in mathematical education-, Toyo-kan's publisher, (4), 303-320. (in Japanese)

Perkinson, H. J, (2000). Learning from Our Mistakes: A Reinterpretation of Twentieth- Century Educational Theory. keiso shobo, 305. (in Japanese)

Tomoto, T., Imai, I., Horiguchi, T., & Hirashima, T. (2013). A support Environment for Learning of Class Structure by Concept Mapping Using Error-Visualization. *Transactions of Japanese Society for Information and Systems in Education*, 30(1), 42-53.

Van Essen, G, & Hamaker. (1990). The Journal of Educationai Research. *Using Self-Generated Drawings to Solve Arithmetic Word Problems, 83*(6), 301-312.