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'Touchy-Feely Vectors' changes students' understanding and modes of reasoning

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Abstract: Studies have found that student understanding of abstract topics such as vectors, critical to learning physics and its application areas, is limited. Students appear to be relatively fluent with the algebra of vectors, but lack geometric imagination. One possible reason for this could be the paper-based medium currently used to learn and reason about vectors. We use an interactive computational system named TFV (Touchy Feely Vectors), which connects the algebra and the geometric representations involved in vectors, and allows students to learn addition and resolution of vectors in an interactive way. We report results from a short intervention based on a series of tasks designed using TFV, where students explored the conceptual possibilities of the interactive system. We examine in detail two student cases of conceptual change based on the system, illustrating how interaction with such enactive media could restructure complex concepts. These case studies could provide us insights into the mechanisms of learning of abstract concepts afforded by computers and other digital media and provide educational researchers a base to design and develop better curricular and pedagogical interventions.

Keywords: Computational Media; Math education; Conceptual understanding; Vectors; Model-based reasoning

1. Introduction

Understanding vectors is critical while learning topics in physics such as mechanics (White 1983; Aguirre and Rankin 1989; Ortiz 2002), electricity and magnetism (Pepper et al. 2012) as well as engineering. Studies have found that students from junior college to the postgraduate level have shortcomings in the understanding of vectors. Students also find it difficult to apply vectors to solve problems (Flores, Kanim, and Kautz 2004; Nguyen and Meltzer 2003; Knight 1995; Aguirre and Rankin 1989; Aguirre 1988). We argue possibility of the limitations of the paper-based medium affecting imagination (understanding and reasoning) of vectors, and explore the possibilities offered by a digital medium while learning vectors. We do this by using a digital media based system called Touchy-feely vectors (TFV-1, described in the section 2) developed to address the vector learning problem. We report in section 3, two detailed case studies of changes in students understanding and reasoning in the context of vectors after interacting with the system.

1.1 Nature of students' existing reasoning using vectors

Understanding vectors involves learning both the geometric and algebraic modes of representation and operations. For example, a vector can be represented as a mathematical entity in the form of an arrow head and a line denoting the magnitude and the direction, and it can be represented using the rectangular components with respect to a frame of reference (the i,j,k components). All the operations, such as addition, scalar and dot products, have formal procedures and algorithms using both the forms viz., magnitude and direction as well as the rectangular components form. Mathematically it is identical to the translation between the polar and cartesian coordinates that is common in coordinate geometry and with complex numbers (in 2D). This translation, and the resulting equivalence of the two forms of the same vector, appears not to be understood by students. The process of resolution of vectors requires fluency in understanding geometry of right triangle and circle, and trigonometry which students appear to struggle with (Byers 2010; Gur 2009; Orhun

2004). Further, vector is a new mathematical entity for early college students, who are only familiar working with scalars, which are simple magnitudes. Our discussions with physics and mathematics teachers at the junior college level confirm the findings in the literature (Knight 1995; Aguirre 1988; Usharani and Meera 2018), that students struggle with incorporating the notion of direction in their mathematical understanding. A similar struggle occurs in the transition from numerical arithmetic to symbolic algebra. As a result of the complexities involved in understanding and applying vectors effectively, students tend to rely on rote methods, performing mechanical algebraic manipulations using the i,j,k components (Knight 1995; Aguirre 1988; Usharani and Meera 2018) with very little geometric imagination and reasoning. The fact that the algebra of addition of vectors in the i,j,k form simply follows the 'addition' of like terms used in scalar algebra reinforces this tendency further, and habituates students to add vectors using the i,j,k components. This approach leaves them vulnerable when problems require reasoning using geometry.

1.2 Limitations of the paper-based medium

Ong (2013) examines changes in the nature of epistemic activities, and its ripples in the structure of the society, as humanity moved from oral to literate media. He illustrates clearly how the medium of representation changed cognitive processes. The medium of representation, and how the media enables/restricts the handling of representations, dictates the nature of concepts and ideas that we can hold and process. Oral cultures had to create structured patterns of words to ensure that the idea is not lost in oral transmission from person to person. This led to rhythmic and meter scales for poetry, and qualifying nouns with elaborate adjectives adding a lot of 'weight' to the ideas. Once writing technologies were available, Ong argues, the nature of thinking and the levels of abstraction jumped ahead in leaps and bounds. One could externally represent an idea as a written sentence, which one could then build further on, in collaboration with others. The emergence of modern science has been attributed to these affordances of the written medium (Hestenes 2006). Extending this view further, a recent work (de Freitas 2016) argues that computational media presents a similar shift, where new media allows new thinking processes, by helping link cognitive and mental processes to bodily states and actions.

An analysis of two popular Indian textbooks and their presentation of topics related to vectors revealed possible learning problems connected to the medium (Karnam and Sule 2018). The analysis showed very fragile interlinking between topics related to vectors. The topics related to vectors were spread across almost 4 grades in general science/ physics and mathematics. In addition, the way the school system is organized (with different teachers in different grades, and a shift in the schooling level after grade 10, requiring going to a different school), a lack of continuity is inherent in the system, which prevents scaffolding the conceptual jumps required to understand the topic. This problem can be attributed at one level to the paper-based medium, where complex content can only be presented in a modular and linear fashion, and classes and grades are organized around this modular structure. Further, textbooks at the beginner levels (grade 11 in India) had very few problems requiring the geometric mode of reasoning, such as adding vectors using triangle and parallelogram laws. This leads to students using more algebraic modes, particularly if there is no serious effort by teachers to address this limitation. The teachers are also limited by the blackboard as the medium of teaching. They expressed their inability in enacting (Rahaman et al. 2017) vector-related topics, as this required drawing many diagrams to make students imagine the dynamics underlying the vector representations. This is also an extended effect of the written medium, where performing geometric manipulations is tedious (requiring usage of geometrical instruments like compass, ruler, protractors, etc). These factors together account for the students' reliance on algebraic reasoning with limited geometric thinking.

2. Touchy-Feely vectors (TFV) - an enactive system to learn Vectors

To address these and other limitations of the paper-based media (such as limited interactivity and collaboration), a new media system is developed that allowed both students and teachers to create vectors and actively manipulate them. There could be already existing systems (some dynamic geometric software or geometric calculators) which could have met some of the design requirements but not all of them. But there are certain other key differences and limitations in certain aspects and given the scope and focus of the paper, we deem those discussions to be irrelevant. Further, given a larger interest in understanding the learning mechanisms, we would need more control on the design of the system and chose to develop the system, ourselves.

A brief description of the system is provided below. For detailed descriptions and design decisions, please see (Karnam et al. 2016). The browser-based interface allows students to create, manipulate, add and resolve vectors dynamically. The system seeks to integrate the algebra and geometry understanding of vectors, and uses a circle (similar to a unit circle) as an interface element to do this. It builds on students' existing understanding of geometry and trigonometry of right triangle and the circle. A click on the screen creates a vector. Holding the line and dragging allows changing the direction. By clicking inside the circle and dragging the magnitude can be changed. These operations are reflected simultaneously in the numerical values. A double click on a circle creates a right triangle inscribed in the circle (see vector C in fig.1a). A single click further inside the circle creates the resolved rectangular components and detailed equations (see vector B in fig.1a). The same interactions in reverse order reverse the process of resolution. One can change the direction and magnitude in all the states. One can add two vectors by clicking on both the vectors holding control (Ctrl) key on the keyboard (see fig.1b). The circle that is highlighted shows the vector that can be manipulated. Both the component vectors can be manipulated by switching between them, by right clicking on either of them. The changes in the resultant are displayed. Double clicking shows the underlying right triangles and a further single click makes the rectangular components appear (see fig.1c) similar to the interactions used for resolution. In all the stages, one can manipulate the vectors' direction and magnitude, and see the effects in the rectangular components as well as the resultant. Returning to the initial triangle law state is through the same interactions of the single-click and double-click. For more details refer to the help menu of the system [link]. The system allows students to see the interconnections between the algebraic and geometrical forms of the vector after resolution, as well as the dynamics behind the process (in short animations). The manipulations on the vector make the concept more tangible. Later versions, based on user feedback from the tests reported here, addresses conceptual limitations, like the lack of the parallelogram law. The new system also has a touch-based interface, which supports embodied learning (Sinclair and de Freitas 2014; de Freitas and Sinclair 2013).

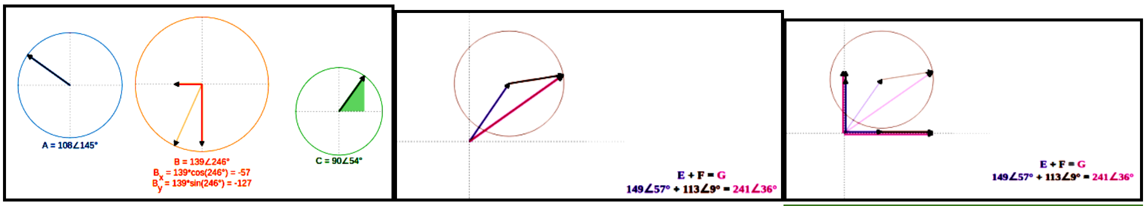

Figure 1(a). Creating a vector, changing its direction and magnitude and resolving it into rectangular components (b). Addition of two vectors (c). Rectangular Components getting added.

3. Case Study

We studied the effect of the system on students' imagination using a series of tasks. Six participants who had finished their grade 11 (finished a yearlong course related to vectors and its applications in physics) were given pre and post tests and were interviewed before and after the TFV intervention. The intervention sessions involved performing tasks on the interactive vector system for about 70-90 minutes. These tasks were designed to make the students explore various features of the system. Their actions were recorded, using video, written scripts (rough work), screen capture, and eye tracking (Tobii X2-60). In an earlier paper in the conference, we have reported the effects that this intervention had in these 6 students post-test performance quantifying the changes in the strengths of conceptual understanding (Karnam et al. 2016). It was reported that the intervention helped in growth in conceptual understanding in most of the conceptual areas. Further, in certain conceptual areas where the pretest strength was good, there was a drop. It was explained as a disruption (sort of a cognitive conflict (Posner et al. 1982; Hewson and A'Beckett Hewson 1984)) created by the intervention in the existing stabilized understanding (perhaps stabilized by rote learning and mass-practice). This state of confusion could allow for better conceptual understanding (D'Mello et al. 2014/2). Now, we will discuss below detailed case studies of two students, capturing their understanding before and after the intervention. These two cases are chosen to illustrate qualitatively a wide range of effects of the intervention (interaction with the TFV system) in the students' conceptual understanding as well as the reasoning modes.

3.1 Case-1 (S2)

S2 has finished grade 11 and gets coached for the entrance exam for IITs (a group of Indian technological institutes, where the admission is based on an entrance exam known to assess students' conceptual understanding). S2's responses in the pretest show hints of conceptual understanding, but he made some procedural mistakes. He showed command over understanding of vector and scalar, correctly identifying the nature of physical quantities like weight, pressure, time. Further, there was a sense of confidence and haste in the way he wrote his answers.

Pretest: Consider S2's response to the question on magnitude of resultant or the dot product (using components) as shown in figure 2. In the problem on the left, he initially has some confusion when dot products need to be calculated using the rectangular components. He starts off using the determinant form (used for the cross product). Even when he rectifies, he directly writes some expression. Similarly, in the problem on the right, he wanted to compare procedurally the magnitudes of resultant and the initial vectors, but he uses the expression of dot product by mistake for the expression of the resultant. This shows that he does not have a strong conceptual understanding, and he has memorized the algebraic expressions (formulae) used in the vectors, possibly just rote learning the procedures. However, he also seems to understand the role of trigonometry in vectors. This understanding is used when he explains how pressure is a scalar, as it does not follow vector addition.

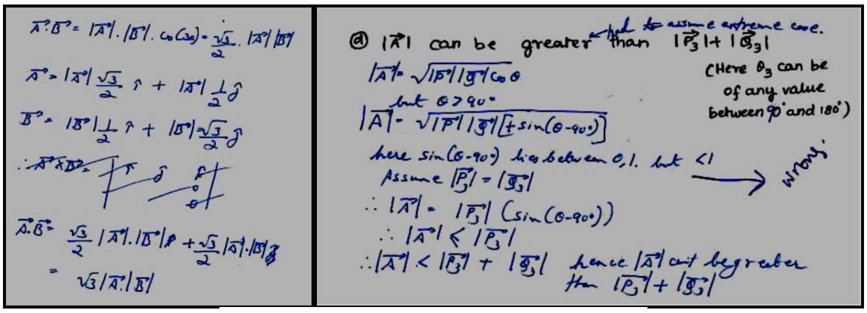

Figure 2. Responses of S2 in Pretest

The interview after the pretest indicated that he had a strong sense of confidence in his conceptual understanding. He started off by dismissing the test as boring. We discussed the question to the right in figure 2 (probing the relation between |A| and |P3|+|Q3, when A=P3+Q3) which he found tricky, as he had to think of extreme cases. In fact, he referred to them as vectors, and not as magnitude of vectors. However, as discussed earlier, he confused the formulae for the resultant and the dot product. Once he had the formula written, he tried to bring in some principles of trigonometric function in the quadrants. He seemed to have a fair overall picture of the concepts involved in trigonometry and its applications in physics. He showed how the rectangular components can add up to give the initial vector, geometrically. He also showed an understanding of vectors as geometrical entities, helping in reasoning about certain problems in physics. He drew examples from resolutions of forces, projectile motion and mathematics (circle and conics). While he did not seem to display a finer understanding in the pretest answer script, he clarified his conceptual stance during the discussion. One of the key corrections was his acknowledgement of non-rectangular components, which he displayed by being careful in using words in the discussion. Though he has breadth in his narrative pertaining trigonometry, this seems to break when probed in detail, as in the case of the question Pr.5d (figure 2 right).

<u>Intervention:</u> S2 was able to complete all the tasks in the intervention session. However, there was a key difference in the way he accomplished the tasks, compared to other students. Different students made use of the TFV system to different extents to help their reasoning. Some relied completely on the geometrical entities on screen to guide their actions towards the goal. Some relied on the algebraic entities on screen to guide their actions. Some tried to bring in their school knowledge. The peculiar thing about S2 was that despite the presence of the interactive system, he relied on pen and paper based calculations to arrive at the solutions to the intervention problems (tasks). He then recreated them on the TFV system. When asked to create a target vector 50î+80ĵ, he made a rough estimate of the resulting magnitude using the formula, and then fine-tuned it to arrive at the target. He followed a similar approach when asked to create a set of two vectors, which add to give a resultant of magnitude 120 at a direction 60° (anti clockwise) with the x axis. He resolved the vector into rectangular components and used them as two vectors that add up to the resultant. This is a very interesting strategy, hinting at conceptual clarity in addition and resolution of vectors. However, this strategy does not cohere well with the confusions in the responses in the pre-test, suggesting that his model-based reasoning is immature. The strategy could be part of one of the broad pictures he has developed, with little connection to the overall vector model. However, further tasks (where he was asked to create other possible vector sets that add up to give the same resultant, with the added constraint that they be perpendicular to each other) demanded exploration of the geometrical affordances of the TFV system. Even in these cases, he tried hard to find ways in which he could use some formulae to arrive at a rough estimate of the required vectors. For instance, in the case of a set of perpendicular components adding up to give a target vector, he assumed them to be of equal magnitude, and then estimated their magnitude as shown in figure 10(a). In the follow-up discussion, when asked about his reasoning using calculations with pen and paper, he called it fundamental understanding, and associated this problem-solving approach with theory, while the interaction with the TFV system was considered an experiment.

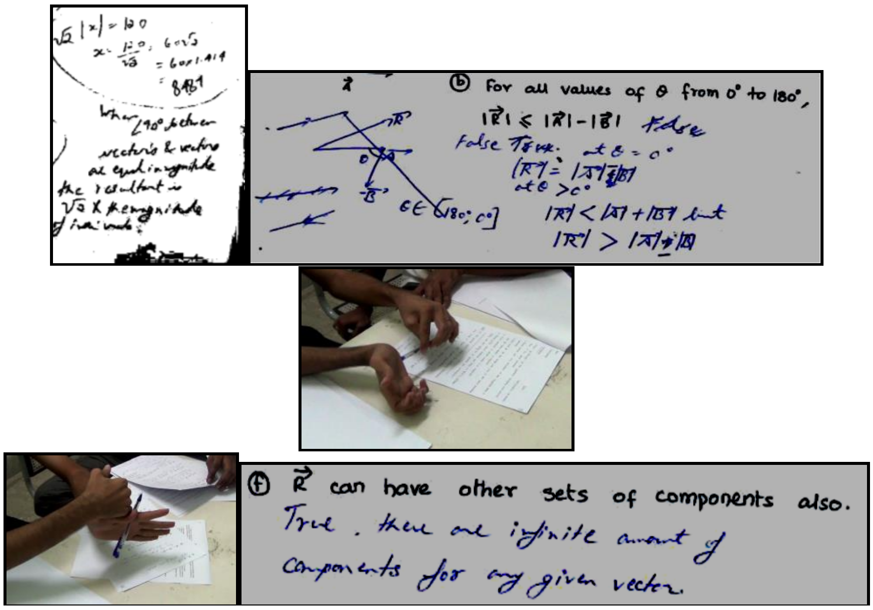

Figure 3(a). A section of the rough work performed by S2 during intervention session. (b) A response by S2 in the post-test. (c) S2 using gestures in the post-test interview to show a dynamic change in the direction of vector. (d) S2 using gestures in the post-test interview to show a how component vectors change to give the target vector. (e) A response by S2 in post-test

Post-test: In the post test, S2 handled a similar problem, as shown in figure 3b, by using diagrams and directly changing the angle dynamically, and by considering the extreme conditions. This clearly shows a fundamental shift in his reasoning approach, from a dominant formula-based algorithmic mode to a more model-based approach. The effect of the system was evident in the way S2 used gestures, as shown in figure 3c, to dynamically change the vectors (Soto-Johnson and Troup 2014; Roth 2003; Roth and Welzel 2001). However, he still articulated this reasoning as being another method/ algorithm, to effectively solve the problems of this kind. To another question, he says- *"...in my mind, I just... I drew two axes and then I shifted the axes like this (gesturing and rotating his hands in figure 3d) ... perpendicular, perpendicular, perpendicular... oh!! There can be many!!*" Later, he explicitly said he realized this when he was performing a task during the intervention session, where he was asked to come up with another set of vectors, which add up to a given target vector. There were many such instances where the post-test task required creating all possible sets of vectors to reach a target. He explicitly mentioned in such cases how the interaction with the system helped him. The post-test revealed the limitations of the student's understanding of components and rectangular components. When solving certain questions on components (both rectangular and non-rectangular), S2 was still looking for formulae, and some confusion made him not attempt the questions at all. The actual intervention data need to be further analyzed (eye tracking data as well as the screen recordings) in detail for a clear trajectory of the changes in S2's understanding.

3.2 Case-2 (S5)

S5 finished her 11th grade from the same college as S2. She takes some remedial help from coaching classes, but was not preparing for the IIT entrance exam. All here responses were neatly and carefully written, and much care was taken to present them as clearly as possible.

Pretest: Her responses in the pretest showed a good understanding of the prerequisites. She could differentiate all physical quantities as vectors and scalars, including pressure, time and friction, correctly, except for weight. One of her mistakes was ignoring the order of vectors in the case of triangle law of vector addition (figure 4a). However, during the interview, it struck her that the vectors need to be ordered in a particular way in triangle law (figure 4b). But, she immediately contradicted herself, as she found the geometrical figures similar (both obtuse triangles). Discussing the questions related to addition of vectors, she could articulate a surface connection (how they related as geometries) between the triangle and parallelogram laws of vector addition. She fumbled while solving questions related to adding using rectangular components. When asked to add a vector after resolving into components, she could resolve them using trigonometry, but could not figure out how to move ahead (figure 4c). The relation between rectangular components and the initial vector was not very clear to her, as evident from the inaccurate representations in the figure (the lengths of the projections made randomly), particularly given her care in writing. When probed about this during the interview, she said that the x component is the projection of the vector on x-axis, and the y-component was similar. But when asked to write the relation between the vector and the rectangular components, she wrote the expression $C\cos\theta + C\sin\theta = 1$, (figure 4d). When prompts were provided using the questionnaire and the textbook, she agreed on $C\cos\theta\hat{i} + C\sin\theta\hat{j} = \mathbf{C}$. This suggests a vague recollection of identities from mathematics relating sin and cosine ratios, but no strong conceptual links in the context of vector components. When probed for general understanding of components, using a set of questions on addition of vectors and triangle law and the components (figure 5a), she was not clear with the diagram, as she expected components to be only x and y components. And her confusion is well articulated in the response she gave to the same set of questions (in figure 5b).

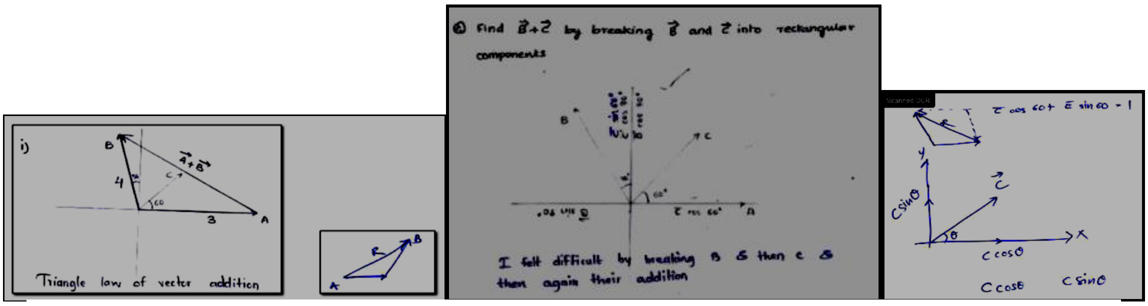

Figure 4. (a) S5's response in a pretest question (improper ordering of vectors triangle law of addition). (b) S5's writing in the pretest interview in the same context (proper ordering of vectors). (c) Response of S5 to a question in the pretest. (d). S5's writings during the pretest interview

In another exercise on rotated frames of reference (figure 6a), she could correctly and confidently represent the x and y components in both the frames of reference. Her understanding of their relation was not clear; as she said the length of x component Acos80° is infinite. Further, the picture she drew (figure 6b), with a right triangle and the sinusoidal wave in the same graph, shows a mixed set of confusions, possibly driven by textbook diagrams. She tries to somehow connect the right triangle narrative of trigonometric ratios with rotating angle and the sinusoidal waveform. This clearly shows a lack of understanding of the unit circle notion.

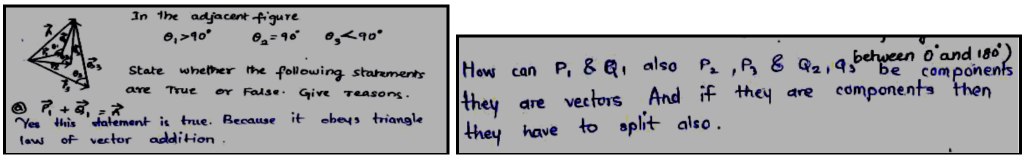

Figure 5(a). A sample question from S5's pretest. (b) S5's response to the set of questions in (a)

Intervention: S5 took a lot of time to get comfortable with the interactions during the intervention session, and could not complete all the tasks within the time available, though she could explore all the features of the system by doing some extra tasks. In the initial tasks of arriving at a target vector such as 50î+100ĵ, she tried changing only the direction of the vector for a long time, and thus could never reach the target. Even when she was given hints to try changing the magnitude, she could not find a systematic way to arrive at the target vector. She was not trying to control both magnitude and direction, and always ended up controlling just one of them. She was able to move ahead only when a case was demonstrated, focused on changing both the direction and magnitude, without discussing the general principle. After this demonstration, she started arriving at the target vectors relatively easily.

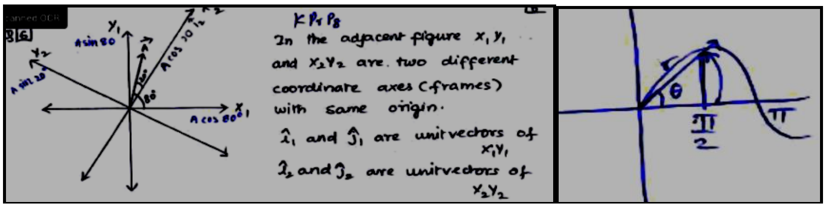

Figure 6 (a) S5’s response on a pretest question during interview. (b) S5’s drawing.

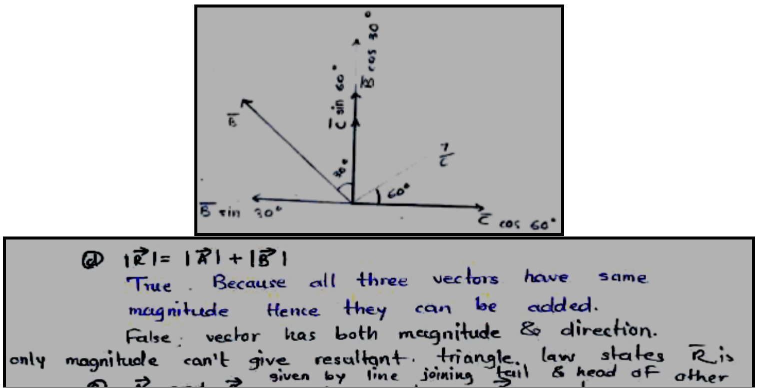

Figure 7 (a) S5’s response in post test. (b) S5’s response in the post-test (blue) and during the post-test interview (black)

She relied completely on the screen controls and did not use any pen and paper calculations. Even during the tasks, where she needed to create a set of vectors that added up to a given target vector, she just kept manipulating the vectors, until she stumbled upon the target vector. During the discussion after the intervention, she said the tasks were simple, and as she kept doing tasks, they seemed simpler. Further, she said with some excitement- “*the tool helped in understanding addition of vectors... hmm... components (meant rectangular components), like the way it is shown here (while drawing). Firstly the vectors, which were inside a circle… Earlier I was not used to how the components come. Like it is shown here (pointing to the system), the lines appear and then this line moves here, forming the y component. Resolution of vectors… umm… angle… I liked this very much.*” Towards the end, as a suggestion, she pointed out that the system now supports only triangle law, but she would like to have parallelogram law as well. She also suggested some ways of implementing it. As an overall comment, she said: *“it (interaction with the system) has made a lot of things easier. As earlier I could not draw things out on my own for addition of vectors and all… angles and others I feel are now easier…”*

Post-test: However, a more complex picture emerged in the post-test and the follow-up discussion. The post-test responses showed a drop in rectangular components. Right in the beginning of the interview, she says that she had confusions on components. She had not properly connected triangle and parallelogram laws, which is of course not expected directly from the intervention. Regarding addition of vectors using rectangular components, she showed some progress from the pretest (see figure 13(a)), where she could not move ahead after resolving the vectors geometrically. In the post test, she said that she could add the y components of both vectors, but was unclear about the x components (as they were in opposite directions in figure 7a). When probed on her responses to questions related to the relations of magnitudes (question in figure 3b), she said she understood the problem when done on the system, but not in the post-test. This clearly indicates a problem of transfer. She made many inconsistent statements as well (figure 7b). But when asked questions explicitly, she could give conclusive answers. She seemed confused on the way the equation **R=A+B** captured both direction and magnitude of the vector. A deeper analysis of her understanding, based on intervention data, is ongoing.

4. Discussion and Conclusions

The two cases illustrate how an interactive media system for learning model-based reasoning could change conceptual understanding in different ways. S2, who relies heavily on algebraic expressions and formulae in the pretest and even during intervention using pen and paper for reasoning, was pushed by some tasks to rely on the system. In the post test, he uses more pictures and gestures, which are indicators of moving towards deeper conceptual understanding and a model-based reasoning. Further, though the post test responses particularly for those related to rectangular components indicate a drop, which indicate a disruption or confusion in his earlier conceptions, and thus provides scope for better learning as can be seen in the interview responses.

Unlike S2, S5, who was not clear on how different concepts were interrelated, started making some conjectures on the basis of working with the system. However, she struggled in transferring this understanding to formal representations. She did show a confidence and an improved handling of the concepts of resolution and addition using triangle law.From a learning sciences perspective, a deeper analysis of the actual interactions of the students during the intervention (analysis of the screen recordings, the video recordings and the eye-tracking data) leading to the conceptual changes will help in two ways. Firstly, it would allow us get a better understanding of the mechanisms involved in the learning of abstract concepts. Secondly, this in turn will put the education researcher community in a better place in designing and establishing better curricular and pedagogical interventions. However, the above cases provide initial promise in the computational system assisting students with widely different conceptual issues, in developing model based reasoning related to vectors. These cases also hint that the intervention, even though limited to a single session, can influence conceptual understanding, and integrate algebraic and geometric thinking. But, for the disruptions to settle, leading to better conceptual understanding, a more sustained interaction for longer duration would be recommended. This would require a proper integration of such systems with the conventional text-book, and teaching based on the textbook, along with sustained interaction within the existing classrooms. Our ongoing work is in this direction.

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